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AN APPROXIMATE SOLUTION OF THE AXISYMMETRIC CONTACT PROBLEM FOR AN ELASTIC SPHERE†

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An axisymmetric, fractionally non-linear contact problem for an elastic sphere with a priori unknown boundary of the contact area is considered. An integral equation for determining the density of the contact pressures is constructed taking account of the shear displacements of the boundary points of the elastic body. An approximate solution, which refines the equations of Hertz' theory, is constructed in the case of a small contact area.

1. FORMULATION OF THE PROBLEM

Consider the axisymmetric problem of the indentation of an elastic sphere $r \le R$ into a rigid support, specified in a spherical system of coordinates (r, θ, φ) by the equation

$$r = R[1+\rho(\theta)], \quad \rho(0) = 0, \quad \rho'(0) = 0 \tag{1.1}$$

It is assumed that the surface of the sphere is free from shear forces and that the sphere is deformed under the action of a normal load

$$\sigma_r = Q(\theta), \quad \gamma_0 \le \theta \le \pi; \quad \sigma_r = -p(\theta), \quad 0 \le \theta \le \gamma \tag{1.2}$$

Here, $Q(\theta)$ is given, $p(\theta)$ is the required pressure, the circle $\theta = \gamma$ on the sphere $r = R$ bounds the contact area, which is unknown a priori and $\gamma < \gamma_0$.

The equation for the static equilibrium of the sphere has the form

$$2\pi R^2 \int_0^\gamma p(\alpha)\sin\alpha\cos\alpha d\alpha = P \tag{1.3}$$

where P is the resultant force of the external pressure $Q(\theta)$.

We will denote the convergence of the centre of the sphere with the support by δ_0 and the elastic displacements of the surface points by $u_r(\theta)$ and $u_\theta(\theta)$. Then, as a result of the deformation of the sphere, a point with coordinates (R, θ, φ) receives radial $u_r(\theta) + \delta_0\cos\theta$ and shear $u_\theta(\theta) + \delta_0\sin\theta$ displacements. The spherical coordinates of the point being considered in the deformed state (subject to the condition that the quantities $u_r(\theta)$, $u_\theta(\theta)$ and δ_0 are small compared with the radius R) will be

$$(R + u_r(\theta) + \delta_0\cos\theta, \theta + R^{-1}(u_\theta(\theta) - \delta_0\sin\theta), \varphi) \tag{1.4}$$

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We will now assume that the point being considered has come into contact with the fixed support. On substituting the coordinates of the point (1.4) into the equation of the support (1.1), we find the contact condition in the form

$$R + u_r + \delta_0\cos\theta = R[1 + \rho(\theta + R^{-1}(u_\theta - \delta_0\sin\theta))], \quad 0 \le \theta \le \gamma \tag{1.5}$$

From relation (1.5), we derive the linearized contact condition, taking account of the shear displacements

$$R + u_r + \delta_0\cos\theta = R[1 + \rho(\theta) + \rho'(\theta)R^{-1}(u_\theta - \delta_0\sin\theta)], \quad 0 \le \theta \le \gamma \tag{1.6}$$

Finally, neglecting the third term in the square brackets on the right-hand side of Eq. (1.6), we obtain [1]

$$R + u_r + \delta_0\cos\theta = R[1 + \rho(\theta)], \quad 0 \le \theta \le \gamma \tag{1.7}$$

In accordance with the solution of the axisymmetric problem of the loading of an elastic sphere by a normal load at its surface $\sigma_r = N(\theta)$ when $0 \le \theta \le \pi$ obtained in [2, 3], the radial and shear displacements of the surface points are represented by the integrals

$$u_r(\theta) = \frac{R}{2\pi G}\int_0^\pi N(\alpha)H_r(\theta, \alpha)\sin\alpha d\alpha \tag{1.8}$$

$$u_\theta(\theta) = \frac{R}{2\pi G}\int_0^\pi N(\alpha)H_\theta(\theta, \alpha)\sin\alpha d\alpha \tag{1.9}$$

Here,

$$H_r(\theta, \alpha) = \frac{\pi}{2}\frac{1-2\nu}{1+\nu} + 4(1-\nu)U(1, \theta, \alpha) + \mathrm{Re}\int_0^1\left(\frac{A_r}{y^m} + \frac{1}{y^2}\right)U(y, \theta, \alpha)dy \tag{1.10}$$

$$H_\theta(\theta, \alpha) = \frac{\partial}{\partial\theta}\mathrm{Re}\int_0^1\left(\frac{A_\theta}{y^m} + \frac{1}{y^2}\right)U(y, \theta, \alpha)dy \tag{1.11}$$

G is the shear modulus and ν is Poisson's ratio.

The function U, which appears in the kernel of (1.10) and (1.11), is expressed in terms of a complete elliptic integral of the first kind $\mathbf{K}(k)$ and has the following form

$$\begin{gathered} U(y, \theta, \alpha) = \frac{\mathbf{K}(k)}{k} - \frac{\pi}{2}(1 + y\cos\theta\cos\alpha) \\ h^2 = (1-y)^2 + 4y\sin^2\frac{\theta+\alpha}{2}, \quad k^2h^2 = 4y\sin\theta\sin\alpha \end{gathered} \tag{1.12}$$

The constants A_r, A_θ and m depend solely on Poisson's ratio:

$$A_r = 8\nu^2 - 8\nu + 1 + i\frac{16\nu^3 - 16\nu^2 - 4\nu + 5}{\sqrt{3 - 4\nu^2}} \tag{1.13}$$

$$A_\theta = 4\nu - 3 + i\frac{18\nu^2 + \nu - 2}{2\sqrt{3 - 4\nu^2}}, \quad 2m = 1 - 2\nu + i\sqrt{3 - 4\nu^2} \tag{1.14}$$

Substituting expressions (1.8) and (1.9) into the refined contact condition (1.6) we obtain an integral equation in the contact pressure

$$-\frac{R}{2\pi G}\int_0^{\gamma} p(\alpha)H(\theta,\alpha)\sin\alpha d\alpha = \upsilon(\theta), \quad 0\le\theta<\gamma \tag{1.15}$$

The function

$$\upsilon(\theta) = R\rho(\theta) - \delta_0\cos\theta - \rho'(\theta)\delta_0\sin\theta - \frac{R}{2\pi G}\int_{\gamma_0}^{\pi} Q(\alpha)H(\theta,\alpha)\sin\alpha d\alpha \tag{1.16}$$

which is defined, apart from the quantity δ_0, is denoted by $\upsilon(\theta)$.

The kernel of the integral operator (1.15) is expressed in terms of the kernels (1.10) and (1.11) as

$$H(\theta,\alpha) = H_r(\theta,\alpha) - \rho'(\theta)H_\theta(\theta,\alpha) \tag{1.17}$$

In the case of a concave spherical support of radius $R_1 > R$

$$R\rho(\theta) = \sqrt{R(2R_1-R)+(R_1-R)^2\cos^2\theta} - (R_1-R)\cos\theta - R \tag{1.18}$$

In the case of a convex spherical support of radius R_1, R has to be replaced by $-R$ on both sides of equality (1.18). In the case of a plane support, in the limit with respect to R_1, we have

$$\rho(\theta) = (\cos\theta)^{-1} - 1 \tag{1.19}$$

The contact problem for an elastic sphere with a fixed contact area (a support with a sharp edge) has been treated by different methods in [4, 6, 7]. Contact problems with an interface of the boundary conditions, unknown in advance, have been investigated on the basis of the contact condition (1.7) in [1, 7] using the method in [8] (see also [9, Section 55]). The method in [10] is used below to construct an approximate analytical solution in closed form. Three-dimensional contact problems for an elastic half-space in a refined formulation, which takes account of the shear displacements, have been considered earlier in [11, 12].

2. REPLACEMENT OF VARIABLES

We will put

$$\operatorname{tg}\frac{\theta}{2} = \varepsilon x, \quad \operatorname{tg}\frac{\alpha}{2} = \varepsilon t, \quad \operatorname{tg}\frac{\gamma}{2} = \varepsilon \tag{2.1}$$

and, in addition, we will introduce the following notation, which is analogous to that adopted earlier in [1],

$$q(x) = \frac{4p(2\operatorname{arctg}\varepsilon x)}{G(1+\varepsilon^2x^2)^{3/2}}, \quad w(x) = -\frac{2\upsilon(2\operatorname{arctg}\varepsilon x)}{R(1+\varepsilon^2x^2)^{1/2}} \tag{2.2}$$

$$S_r(x,t) = \frac{t}{\sqrt{(1+\varepsilon^2x^2)(1+\varepsilon^2t^2)}}\left[\frac{1}{\pi}\operatorname{Re}\int_0^1\left(\frac{A_r}{y^m}+\frac{1}{y^2}\right)U^0(y)dy + \right.$$

$$\left. + \frac{1-2\nu}{2(1+\nu)} - 2(1-\nu)\left(1+\frac{(1-\varepsilon^2x^2)(1-\varepsilon^2t^2)}{(1+\varepsilon^2x^2)(1+\varepsilon^2t^2)}\right)\right] \tag{2.3}$$

$$S_\theta(x,t) = \frac{1}{2\varepsilon\pi}\frac{t(1+\varepsilon^2x^2)}{\sqrt{(1+\varepsilon^2x^2)(1+\varepsilon^2t^2)}}\frac{\partial}{\partial x}\mathrm{Re}\int_0^1\left(\frac{A_\theta}{y^m}+\frac{1}{y^2}\right)U^0(y)dy$$
$$\theta_1 = \frac{1-\nu}{2\pi},\quad U^0(y) = U(y, 2\,\mathrm{arctg}\,\varepsilon x, 2\,\mathrm{arctg}\,\varepsilon t) \tag{2.4}$$

After making the replacement of variables (2.1), we can write Eq. (1.5), taking account of expression (1.17), in the form

$$\frac{\theta_1}{\varepsilon}\int_0^1 q(t)\frac{4t}{x+t}\mathbf{K}\left(\frac{2\sqrt{xt}}{x+t}\right)dt = -\int_0^1 q(t)S(x,t)dt + \frac{1}{\varepsilon^2}w(x) \tag{2.5}$$

$$S(x,t) = S_r(x,t) - \rho'(2\,\mathrm{arctg}\,\varepsilon x)S_\theta(x,t) \tag{2.6}$$

At the same time, the static equilibrium equation (1.3) is rewritten as

$$\varepsilon^2\int_0^1 q(t)\frac{(1-\varepsilon^2t^2)t}{(1+\varepsilon^2t^2)^{3/2}}dt = \frac{P}{2\pi R^2G} \tag{2.7}$$

Hence, as a result of these transformations in the initial integral equation, the integral operator corresponding the axisymmetric contact problem for an elastic half-space is separated out in explicit form.

3. THE METHOD FOR THE APPROXIMATE SOLUTION OF THE REFINED AXISYMMETRIC CONTACT PROBLEM

We will make use of the previously obtained [13–15] general solution

$$q(x) = \frac{F(1)}{\pi\sqrt{1-x^2}} - \frac{1}{\pi}\int_x^1\frac{F'(s)}{\sqrt{s^2-x^2}}ds \tag{3.1}$$

$$\pi F(x) = u(0) + x\int_0^x\frac{u'(z)}{\sqrt{x^2-z^2}}dz \tag{3.2}$$

of the integral equation of the axisymmetric contact problem for an elastic half-space

$$\int_0^1 q(t)\frac{4t}{x+t}\mathbf{K}\left(\frac{2\sqrt{xt}}{x+t}\right)dt = u(x)$$

We substitute the expression

$$\frac{\theta_1}{\varepsilon}u(x) = \frac{1}{\varepsilon^2}w(x) - \int_0^1 q(t)S(x,t)dt \tag{3.3}$$

into formula (3.2). We obtain

$$\frac{\pi\theta_1}{\varepsilon}F(x) = \frac{1}{\varepsilon^2}w(0) - \int_0^1 q(t)S(0,t)dt + \frac{x}{\varepsilon^2}\int_0^x\frac{w'(z)dz}{\sqrt{x^2-z^2}} - \int_0^1 q(t)x\int_0^x\frac{S'_z(z,t)dz}{\sqrt{x^2-z^2}}dt$$

and, after some reduction

$$\frac{\pi\theta_1}{\varepsilon}F(x) = \frac{1}{\varepsilon^2}w(0) + \frac{x}{\varepsilon^2}\int_0^x \frac{w'(z)dz}{\sqrt{x^2 - z^2}} - \int_0^1 q(t)\left(S(x, t) - x\int_0^x \frac{S(z, t) - S(x, t)}{(x^2 - z^2)^{3/2}}zdz\right)dt \tag{3.4}$$

From the condition that the contact pressure vanishes on the boundary of the contact area, we derive the equality $F(1) = 0$. According to expression (3.4), we arrive at the equation

$$w(0) + \int_0^1 \frac{w'(z)dz}{\sqrt{1 - z^2}} - \varepsilon^2\int_0^1 q(t)\left(S(1, t) - \int_0^1 \frac{S(z, t) - S(1, t)}{(1 - z^2)^{3/2}}zdz\right)dt = 0 \tag{3.5}$$

Equation (3.5) serves to determine the required angle γ and the coordinates of the boundary of the contact area.

By virtue of the equality $F(1) = 0$, formula (3.1) can be rewritten as

$$q(x) = -\frac{1}{\pi}\int_x^1 \frac{F'(s)}{\sqrt{s^2 - x^2}}ds \tag{3.6}$$

Substituting expression (3.6) into formula (2.7), after changing the order of integration, we obtain

$$\frac{1}{\varepsilon^2}\frac{P}{2\pi R^2 G} = -\frac{1}{\pi}\int_0^1 F'(s)ds\int_0^s \frac{(1 - \varepsilon^2 t^2)tdt}{\sqrt{s^2 - t^2}(1 + \varepsilon^2 t^2)^{3/2}}$$

We now evaluate the internal integral and then integrate by parts. As a results we obtain

$$\frac{1}{\varepsilon^2}\frac{P}{2\pi R^2 G} = \frac{1}{\pi}\int_0^1 F(s)\left[\frac{1 - \varepsilon^2 s^2}{1 + \varepsilon^2 s^2} - \frac{2\varepsilon s}{(1 + \varepsilon^2 s^2)^{3/2}}\ln(\sqrt{1 + \varepsilon^2 s^2} + \varepsilon s)\right]ds \tag{3.7}$$

From formula (3.6), we derive the following expression for the maximum of the contact pressures (at the pole of the sphere)

$$q(0) = -\frac{1}{\pi}\int_0^1 \frac{F'(s)}{s}ds \tag{3.8}$$

Now, integrating by parts, taking account of the values $F(1) = 0$ and $F'(0) = 0$ and substituting the result obtained into expression (3.4), we get

$$\begin{aligned} \frac{\pi^2\theta_1}{\varepsilon}q(0) &= \frac{1}{\varepsilon^2}w(0) - \frac{1}{\varepsilon^2}\int_0^1 \frac{w'(z)}{z}\left(\frac{\pi}{2} - \arcsin z\right)dz - \int_0^1 q(t)S(0, t)dt + \\ &+ \int_0^1 q(t)\int_0^1\left(\frac{S(s, t) - S(0, t)}{s^2} - \frac{1}{s}\int_0^s \frac{S(z, t) - S(s, t)}{(s^2 - z^2)^{3/2}}zdz\right)dsdt \end{aligned} \tag{3.9}$$

According to Hertz' theory [16] (see also [2, 17]) the contact pressure is distributed over the contact area semiellipsoidally

$$q(x) = q_0\sqrt{1 - x^2} \tag{3.10}$$

where q_0 is the maximum value of the contact pressures.

Substituting expression (3.10) into the right-hand side of Eq. (3.9), we obtain an approximate equation for finding the magnitude of q_0. After determining the magnitude of q_0, we substitute expression (3.10) into Eq. (3.5) in order to express the angular coordinate γ of the boundary of the contact area in terms of the quantity δ_0, representing the convergence of the sphere to the support. Finally, substituting expression (3.10) into formula (3.7) we can establish the approximate dependence of the quantity δ_0 on the magnitude of the force P pressing the sphere onto the support.

4. SEPARATING OUT THE SINGULARITIES IN THE FUNCTION $S(x, t)$

Instead of formulae (1.12) for the kernel in the integrals (1.10) and (1.11), we shall use the following representation [3]

$$U(y, \theta, \alpha) = \int_0^{\pi/2} [h_\lambda(y)^{-1} - 1 - y\lambda] d\psi \tag{4.1}$$

$$h_\lambda(y) = \sqrt{y^2 - 2y\lambda + 1}, \quad \lambda = \cos(\theta + \alpha) + 2\sin\theta\sin\alpha\sin^2\psi \tag{4.2}$$

Substituting expression (4.1) into formulae (1.10) and (1.11) we obtain the integral

$$I_\lambda = \int_0^1 \left(\frac{A}{y^m} + \frac{1}{y^2}\right)[h_\lambda(y)^{-1} - 1 - y\lambda] dy \tag{4.3}$$

We will now find the asymptotic form of the integral (4.3) when $\lambda \to 1$. We have

$$I_\lambda = \int_0^1 (Ay^{2-m} + 1)\frac{2 - y^2 + (1 - \lambda^2)(y^2 - 2\lambda y - 3)}{h_\lambda(y)[1 + (1 + y\lambda)h_\lambda(y)]} dy$$

We will initially consider the behaviour of the integral

$$I_\lambda^0 = \int_0^1 (Ay^{2-m} + 1)\frac{dy}{h_\lambda(y)}$$

when $\lambda \to 1$. On separating out the integrals which converge in the limit (when $\lambda = 1$) (in particular, see [18, Chapter 1, Section 4]), we obtain

$$I_\lambda^0 = (A + 1)\int_0^1 \frac{dy}{h_\lambda(y)} - A\int_0^1 \frac{1 - y^{2-m}}{1 - y} dy + 2A(1 - \lambda)\int_0^1 \frac{1 - y^{2-m}}{1 - y}\frac{y dy}{h_\lambda(y)(1 - y + h_\lambda(y))}$$

Now, using the standard integrals

$$I_\lambda^1 = \int_0^1 \frac{dy}{h_\lambda(y)} = -\frac{1}{2}\ln(1 - \lambda) + \ln(\sqrt{2} + \sqrt{1 - \lambda})$$

$$I_\lambda^2 = \int_0^1 \frac{dy}{h_\lambda(y)(1 - y + h_\lambda(y))} = \frac{1}{1 - \lambda}\ln\left(1 + \sqrt{\frac{1 - \lambda}{2}}\right)$$

it can be proved that the following relations hold

$$I_\lambda^0 = \frac{A + 1}{2}(-\ln(1 - \lambda) + \ln 2) - A\int_0^1 \frac{1 - y^{2-m}}{1 - y} dy + O(\sqrt{1 - \lambda}) \tag{4.4}$$

$$I_\lambda = I_\lambda^0 + O(\sqrt{1 - \lambda}), \quad \lambda \to 0 \tag{4.5}$$

Further on making the replacement of variables (2.1) in formula (4.2), we obtain

$$\frac{1-\lambda}{2\varepsilon^2} = \frac{x^2+t^2+2xt\cos 2\psi}{(1+\varepsilon^2x^2)(1+\varepsilon^2t^2)}$$

Bearing in mind the integration in formula (4.1), we evaluate the integral

$$I(\beta) = \int_0^{\pi/2} \ln(1+\beta^2+2\beta\cos 2\psi)d\psi \ (0 \le \beta < 1)$$

Differentiating with respect to the parameter β, we will have

$$\frac{dI}{d\beta} = \int_0^{\pi} \frac{\beta+\cos\psi}{1+\beta^2+2\beta\cos\psi}d\psi = 0$$

By virtue of the initial condition $I(0) = 0$, we find that $I(\beta) \equiv 0$. Correspondingly, we obtain

$$\int_0^{\pi/2} \ln(x^2+t^2+2xt\cos 2\psi)d\psi = \pi\ln\max\{x, t\} \ (x \ne t)$$

Hence, on the basis of the asymptotic formulae (4.4) and (4.5), we establish the relations

$$S_r(x, t) = t\left[-\frac{1}{2}\mathrm{Re}(A_r+1)\ln\varepsilon\max\{x, t\}+c_0\right]+O(\varepsilon), \quad \varepsilon \to 0 \tag{4.6}$$

$$c_0 = \frac{1-2\nu}{2(1+\nu)}-4(1-\nu)-\frac{1}{2}\mathrm{Re}A_r\int_0^1\frac{1-y^{2-m}}{1-y}dy \tag{4.7}$$

We emphasize that formula (4.6) agrees with the known results in [1].

Similarly, we find

$$S_\theta(x, t) = -\frac{\mathrm{Re}(A_\theta+1)}{4\varepsilon}\frac{t}{x}h(x-t)+O(1), \quad \varepsilon \to 0 \tag{4.8}$$

where $h(t)$ is the Heaviside function.

According to the assumptions underlying (1.1), the expansion

$$\rho(\theta) = \alpha_0\theta^2+O(\theta^3), \quad \theta \to 0 \tag{4.9}$$

holds. The quantity $R(1-2\alpha_0)^{-1}$ has the meaning of the radius of curvature of the support at the pole.

Substituting the asymptotic expressions (4.6), (4.8) and (4.9) into formula (2.6), we obtain

$$S(x, t) = t[-\beta^2\ln\varepsilon\max\{x, t\}+c_0]-2\beta\alpha_0 th(x-t)+O(\varepsilon), \quad \varepsilon \to 0 \tag{4.10}$$

The constant c_0 is given by formula (4.7) and, in addition, we have introduced the notation

$$\beta = 1-2\nu \tag{4.11}$$

In deriving relation (4.10), we have taken account of the equalities $\mathrm{Re}(A_r+1) = 2\beta^2$ and $\mathrm{R}(A_\theta+1) = -2\beta$ (see formulae (1.13) and (1.14)).

5. THE CASE OF A SMALL CONTACT AREA

We will now find the expansion of the function $w(x)$, which is defined by the second formula of (2.2) and appears on the right-hand side of Eq. (2.5). For this purpose, we consider the function

$$V(\theta) = -\frac{R}{2\pi G}\int_{\gamma_0}^{\pi} Q(\alpha)H(\theta,\alpha)\sin\alpha d\alpha \tag{5.1}$$

When $\gamma_0 > 0$, the function $V(2\operatorname{arctg}\varepsilon x)$ depends regularly on the parameter ε. In view of the complexity of the resulting formulae, we shall only discuss the special case of the loading of a sphere with a concentrated force at its pole $\alpha = \pi$. At the same time, according to the equilibrium equation (1.3), the function (5.1) takes the form

$$V(\theta) = \frac{R}{2\pi G}\frac{P}{2\pi R^2}H(\theta,\pi) \tag{5.2}$$

It can be shown that the expansion

$$\begin{aligned}\frac{2}{\pi}H(2\operatorname{arctg}\varepsilon x,\pi) &= \frac{1-2\nu}{1+\nu} + 2(1-\nu) + \ln 2 + C_r^1 + \\ &+ \varepsilon^2 x^2\left[-7(1-\nu) + \frac{1}{4} + 2\ln 2 + 2C_r^2 + 8\alpha_0\left(\frac{1}{8} + \ln 2 + C_\theta^2\right)\right] + O(\varepsilon^3)\end{aligned} \tag{5.3}$$

holds when $\varepsilon \to 0$, where the notation

$$C_r^1 = \operatorname{Re}\int_0^1 \frac{A_r y^{2-m}}{1+y}dy, \quad C^2 = \operatorname{Re}\int_0^1 \frac{Ay^{2-m}}{(1+y)^3}(3+3y+y^2)dy$$

has been introduced.

Hence, in the case of (5.2), the following expansion for the function

$$w(x) = -\frac{2}{R(1+\varepsilon^2x^2)^{1/2}}[R\rho(\theta) - \delta_0\cos\theta - \rho'(\theta)\delta_0\sin\theta + V(\theta)]|_{\theta = 2\operatorname{arctg}\varepsilon x}$$

holds on the basis of the equality (5.3)

$$w(x) = w(0) - a_0\varepsilon^2x^2 + O(\varepsilon^3), \quad \varepsilon\to 0 \tag{5.4}$$

$$w(0) = \frac{2\delta_0}{R} - \frac{P}{4\pi R^2 G}\left(\frac{1-2\nu}{1+\nu} + 2(1-\nu) + \ln 2 + C_r^1\right) \tag{5.5}$$

$$\begin{aligned}a_0 &= 8\alpha_0 + (5-16\alpha_0)\frac{\delta_0}{R} + \frac{P}{4\pi R^2 G}\left(\alpha_0(1+8\ln 2) + \frac{1}{4} + \right. \\ &\left. + \frac{3}{2}\ln 2 - \frac{1-2\nu}{2(1+\nu)} - 8(1-\nu) - \frac{1}{2}C_r^1 + 2C_r^2 + 8\alpha_0 C_\theta^2\right)\end{aligned} \tag{5.6}$$

We will now proceed to construct an approximate solution of Eq. (2.5). The scheme which has been described in Section 3 is most easily implemented in the following way. For the density (3.10) we initially calculate the subsidiary functions $u(x)$ and $F(x)$, which are defined by formulae (3.3) and (3.2). So, by expression (4.10), we have

$$\int_0^1\sqrt{1-t^2}S(x,t)dt = (-\beta^2\ln\varepsilon x + c_0 - 2\beta\alpha_0)\int_0^x\sqrt{1-t^2}tdt + \int_x^1\sqrt{1-t^2}t(-\beta^2\ln\varepsilon t + c_0)dt$$

As a result of integration, we obtain

$$\frac{\theta_1}{\varepsilon}u(x) = \frac{1}{\varepsilon^2}w(x) - \frac{q_0}{3}\left\{\beta^2[\sqrt{1-x^2} - \ln(1+\sqrt{1-x^2})] + \right.$$

$$\left. + \left(2\beta\alpha_0 + \frac{\beta^2}{3}\right)(1-x^2)^{3/2} - \beta^2\ln\varepsilon + c_0 - 2\beta\alpha_0\right\} \tag{5.7}$$

Substituting expression (5.7) into formula (3.2) and taking account of the representation (5.4) we obtain

$$\frac{\pi\theta_1}{\varepsilon}F(x) = \frac{1}{\varepsilon^2}w(0) - \frac{q_0}{3}\left(-\beta^2\ln 2\varepsilon + c_0 + \frac{4}{3}\beta^2\right) - 2a_0x^2 +$$

$$+ \frac{q_0 x}{6}\left\{\beta^2\left(\ln\frac{1+x}{1-x} + \frac{1}{x}\ln(1-x^2)\right) + (6\beta\alpha_0 + \beta^2)\left(x + \frac{1}{2}(1-x^2)\ln\frac{1+x}{1-x}\right)\right\} \tag{5.8}$$

On taking the limit on the right-hand side of equality (5.8) as $x \to 1-0$, we find

$$\frac{\pi\theta_1}{\varepsilon}F(1) = \frac{1}{\varepsilon^2}w(0) - 2a_0 - \frac{q_0}{3}\left(-\beta^2\ln 4\varepsilon + c_0 + \frac{5}{6}\beta^2 - 3\beta\alpha_0\right) \tag{5.9}$$

Then, neglecting terms of the order of $O(\varepsilon^2)$ compared with unity in the integrand (3.7), we obtain

$$\frac{1}{\varepsilon^2}\frac{P}{2\pi R^2 G} = \frac{1}{\pi}\int_0 F(s)ds \tag{5.10}$$

Substituting expression (5.8) into formula (5.10), we find

$$\frac{\pi^2\theta_1}{\varepsilon^3}\frac{P}{2\pi R^2 G} = \frac{1}{\varepsilon^2}w(0) - \frac{2}{3}a_0 + \frac{q_0}{3}\left[\beta^2\left(\ln 4\varepsilon - \frac{19}{12}\right) - c_0 + \frac{3}{2}\beta\alpha_0\right] \tag{5.11}$$

To calculate the quantity q_0 according to formula (3.8), we make use of the following representation

$$\frac{\pi\theta_1}{\varepsilon}[F(x) - F(0)] = -2a_0x^2 + \frac{q_0 x}{3}\int_0^x\left(\frac{\beta^2}{1+\sqrt{1-z^2}} + (\beta^2 + 6\beta\alpha_0)\sqrt{1-z^2}\right)\frac{zdz}{\sqrt{x^2-z^2}}$$

Substituting this expression into relation (3.8), we arrive at the equation

$$\frac{\pi^2\theta_1}{\varepsilon}q_0 = \frac{\pi\theta_1}{\varepsilon}F(0) + 2a_0 - \frac{q_0}{3}\int_0^1\frac{ds}{s}\int_0^s\left(\frac{\beta^2}{1+\sqrt{1-z^2}} + (\beta^2 + 6\beta\alpha_0)\sqrt{1-z^2}\right)\frac{zdz}{\sqrt{s^2-z^2}} \tag{5.12}$$

We change the order of integration in the repeated integral in (5.12) and then integrate with respect to the variable s. Finally, using the values of the integrals

$$\int_0^1\sqrt{1-z^2}\arccos z\,dz = \frac{1}{4} + \frac{\pi^2}{16}, \quad \int_0^1\frac{\arccos z}{1+\sqrt{1-z^2}}dz = \frac{\pi^2}{8} - \ln 2$$

we find from Eq. (5.12) that

$$q_0 = \frac{\frac{\varepsilon}{\pi^2\theta_1}\left(\frac{1}{\varepsilon^2}w(0) + 2a_0\right)}{1 + \frac{\varepsilon}{3\pi^2\theta_1}\left[-\beta^2\ln 2\varepsilon + c_0 + \beta^2\left(\frac{19}{12} + \frac{3\pi^2}{16} - \ln 2\right) + 3\beta\alpha_0\left(\frac{1}{2} + \frac{\pi^2}{8}\right)\right]} \tag{5.13}$$

Returning to formula (5.9) and putting $F(1) = 0$, we obtain the equation

$$\frac{1}{\varepsilon^2}w(0) - 2a_0 - \frac{q_0}{3}\left(-\beta^2\ln 4\varepsilon + c_0 + \frac{5}{6}\beta^2 - 3\beta\alpha_0\right) = 0 \tag{5.14}$$

When account is taken of the notation (5.5), the three equations (5.11), (5.13) and (5.14) relate the three unknown quantities ε, δ_0 and q_0. We will now find the asymptotic solution of this system, remaining within the limits of accuracy with which its equations have been derived.

First, eliminating the quantity $w(0)$ from Eqs (5.11) and (5.13), we express the parameter q_0 in terms of the magnitude of the contact force

$$q_0 = \frac{\frac{1}{\varepsilon^2}\frac{P}{2\pi R^2 G} + \frac{8}{3}\frac{\varepsilon}{\pi^2\theta_1}a_0}{1 + \frac{\varepsilon}{\pi^2\theta_1}\left[\frac{\pi^2}{16}\beta^2 + \beta\alpha_0\left(1 + \frac{\pi^2}{8}\right)\right]} \tag{5.15}$$

Second, using Eq. (5.13), we eliminate the quantity $w(0)$ from Eq. (5.14). After this, we substitute the value of the parameter q_0, which is defined by (5.15), into the resulting equation. As a result, we arrive at the equation

$$\frac{4}{3}\frac{\varepsilon^3 a_0}{\pi^2\theta_1} - \frac{\varepsilon^4 a_0}{(\pi^2\theta_1)^2}(\beta^2 + 2\beta\alpha_0) = \frac{P}{2\pi R^2 G} \tag{5.16}$$

Note that, apart from terms of the order of $O(\varepsilon^3)$, Eq. (5.16) agrees with the asymptotic solution using the "large λ" method obtained earlier in [1] (compare with formula (41) of [1] when $\beta\alpha_0 = 0$).

Equation (5.16) serves to determine the required parameter ε, which defines the size of the contact area. The asymptotic solution of Eq. (5.16) when $\varepsilon \ll 1$ (the right-hand side of Eq. (5.16) is assumed to be small) is

$$\varepsilon = \left(\frac{3\pi\theta_1 P}{8a_0 R^2 G}\right)^{1/3}\left[1 + \frac{\beta^2 + 2\beta\alpha_0}{4\pi^2\theta_1}\left(\frac{3\pi\theta_1 P}{8a_0 R^2 G}\right)^{1/3}\right] \tag{5.17}$$

In order to express the approach δ_0 of the centre of the sphere to the support as a function of the force P acting on the sphere, we substitute the expression defined by formulae (5.5), (5.15) and (5.17) into Eq. (5.14). Finally, we will have

$$\frac{\delta_0}{R} = \left(\frac{3\pi\theta_1\sqrt{a_0}P}{8R^2 G}\right)^{2/3} + \frac{P}{8\pi R^2 G}\left[\frac{2}{3}\beta^2\ln\frac{R^2 G a_0}{24\pi\theta_1 P} + C_0\right] \tag{5.18}$$

$$C_0 = \frac{1-2\nu}{1+\nu} + 2(1-\nu) + \ln 2 + C_r^1 + 2c_0 + \frac{19}{6}\beta^2 - 3\beta\alpha_0 \tag{5.19}$$

It should be emphasized that, when $\varepsilon \ll 1$, the asymptotic relations $\delta_0 \sim \varepsilon^2$ and $P \sim \varepsilon^3$ are satisfied. Hence, the quantity a_0 in the resulting equations (5.17) and (5.18), which is defined by formula (5.6), takes the following value

$$a_0 = 8\alpha_0 \tag{5.20}$$

Finally, the approximate expression for the contact pressure density is obtained from formula (3.6) into which it is necessary to substitute expression (5.8), in which the parameter q_0 is given by formula (5.15).

In order to compare the results obtained (5.18) with the analogous result in [1], account has to be taken of the following two facts. First, the integral equation of the contact problem in [1] was constructed without taking account of the shear displacements. Second, the coefficients of the expansion (32) in [1] are determined by the specified external load (see formulae (5.5) and (5.6)). In the special case of (5.2) which is being considered here, this dependence is explicitly separated out and expressed in terms of the magnitude of the contact force P. It is therefore easy to prove that, when the above facts are taken into account, Eq. (5.14) with the substitutions (5.15) and (5.16), from which formula (5.18) was directly obtained, agrees with Eq. (40) in [1], apart from terms of the order of $O(\varepsilon^2)$ compared with unity.

It should be noted that, unlike the approximate solution constructed above, the approximate solution of the contact problem for an elastic sphere in the case of a small contact area proposed in [19] is not asymptotically exact. This means that the error in the solution in [19] turns out to be of the same order as the correction to Hertz' theory derived in [19]. In particular, formula (3.4), which is analogous to formula (5.15), is simply identical to Hertz' formula. At the same time, it is obvious that the solution in [19] does not agree with the asymptotic solution obtained in [1].

6. EXAMPLES

In the case of a concave spherical support of radius $R_1 > R$, we determine $\alpha_0 = (2R_1)^{-1}(R_1 - R)$ from formula (1.8) and, in the case of a convex spherical support of radius R_1, we have $\alpha_0 = (2R_1)^{-1}(R_1 + R)$. In the case of a plane support, formula (1.19) gives $\alpha_0 = 1/2$. Accordingly, the quantity a_0 is calculated using formula (5.20).

Expansion (5.3) was obtained for the case (5.2) of the action of a concentrated external force on an elastic sphere. It is easily seen that only the leading term in expansion (5.3) was required in deriving the resulting asymptotic formulae (5.17) and (5.18). Hence, in the general case (5.1) of the loading of an elastic sphere with a distributed load, formula (5.17) holds good without any changes. However, in formula (5.18), the left-hand side is replaced by the expression $R^{-1}(\delta_0 - V(0))$ and the constant C_0, which is defined by formula (5.19), is replaced by the following: $2c_0 + (19/6)\beta^2$.

We will now briefly consider the problem of the compression of an elastic sphere by two identical punches (see the note in [7]). Separating out any punch, we mainly model the effect on the stressed state in its neighbourhood due to the other punch by the action of a concentrated force. We therefore again arrive at Eq. (5.17) for determining the size of the contact area. In this case, Eq. (5.18) will determine the magnitude of the convergence of the punches, which is equal to $2\delta_0$.

The problem of the compression of two identical elastic spheres by concentrated forces acting along their common axis of symmetry can also be analysed using the solution obtained. This problem reduces to the problem of the pressure of an elastic sphere on a rigid plane support ($a_0 = 4$). The size of the contact area between the elastic spheres is therefore determined by Eq. (5.17) and the convergence of the centres of the spheres, which is equal to $2\delta_0$, is calculated using formula (5.18).

In the case of the compression of two identical elastic spheres by two identical punches, it is necessary to distinguish between the contact area between the punches and the contact area between a sphere and a punch. Accordingly, Eq. (5.17) has to be written twice with different values of the constant a_0. The convergence of the punches is also calculated using double application of formula (5.18).

In the case of the compression of an elastic sphere by two different punches on each of which a force P acts, the radii of the contact areas are determined using formula (5.17). Equation (5.18) with different values of the quantity (5.20) will determine the distance of each of the punches from the centre of the sphere. Hence, the relative convergence of the punches is equal to the sum of the right-hand sides of the above-mentioned equations and their half-difference gives the relative deviation of the centre of the sphere from the mean position in the deformed state.

7. CONCLUSION

It is obvious that the solution of the axisymmetric contact problem which has been obtained for an elastic sphere, largely agrees with Hertz' theory and is an extension of it to the special case being considered.

Note that a rational fractional approximation of the type of (5.13) significantly increases the accuracy of calculations (see [10] and, also, [20, 21]).

We emphasize that the boundary condition (1.7) was assumed on the basis of calculations in [1, 7], which differs from that used in this paper in that there are no terms $\rho'(\theta)u_\theta - \delta_0\rho'(\theta)\sin\theta$. On going through the calculations again (in particular, see formulae (5.6) and (5.20)), we conclude that the resulting asymptotic formulae were obtained neglecting the term $\delta_0\rho'(\theta)\sin\theta$. However, the term $\rho'(\theta)u_\theta$ was used in formulating the kernel (1.17) of the initial integral equation (1.15) and was taken into account in expression (2.6) and the asymptotic representation (4.10) for the kernel $S(x, t)$, which characterizes the difference of an elastic sphere from an elastic half-space. The following "intermediate" version (between formulae (1.6) and (1.7)) of the contact boundary condition can therefore be recommended.

$$R + u_r + \rho'(\theta)u_\theta + \delta_0\cos\theta = R[1 + \rho(\theta)], \quad 0 \le \theta \le \gamma \tag{7.1}$$

In any case ((1.6) or (1.7)), the shear displacements of the boundary points of an elastic body must be taken into account in the refined (differing from the Hertz) formulation of the contact problem.

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